Non-perturbative Determination of Thermal Sommerfeld Effect

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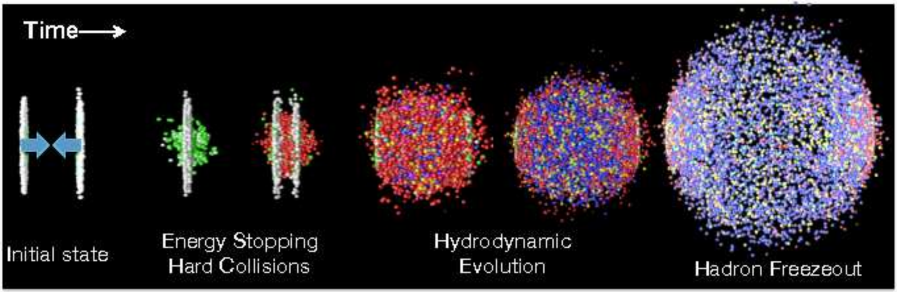
based on
SK, M. Laine (ITP, U of Bern), arXiv:1602.08105

Outline

Thermal Sommerfeld Effect in QGP



Thermal Sommerfeld Effect in QGP

• chemical equilibriation rate of heavy quark in Quark-Gluon Plasma (QGP)

$$\Gamma_{\text{chem}} \simeq \frac{8\pi\alpha_s^2}{3M^2}\left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}\left[\frac{\bar{S}_1}{3}+\left(\frac{5}{6}+Nf\right)\bar{S}_8\right]. \quad (1)$$

Thermal Sommerfeld Effect in QGP

- How to calculate: consider Boltzmann equation

$$\partial_t n \simeq -c(n^2 - n_{eq}^2) = \dot{n}_{\rm loss} + \dot{n}_{\rm gain} \tag{1}$$

with $\dot{n}_{\rm loss} = -cn^2$

- in equilibrium, $n(t) = n_{\rm eq}$ and $\delta\dot{n} \simeq -2cn\delta n$

$$\Gamma_{\rm chem} = \frac{\delta\dot{n}}{n}\Big|_{\rm eq} = -2\frac{\dot{n}_{\rm loss}}{n_{eq}} \tag{2}$$

Thermal Sommerfeld Effect in QGP

- then

$$\Gamma_{\rm chem} = \frac{2}{2N_c \int_{\mathbf{k}} f_F(E_k)} \int\int (2\pi)^4 \delta^4(P_1 + P_2 - K_1 - K_2) f_F(E_{k_1}) f_F(E_{k_2}$$

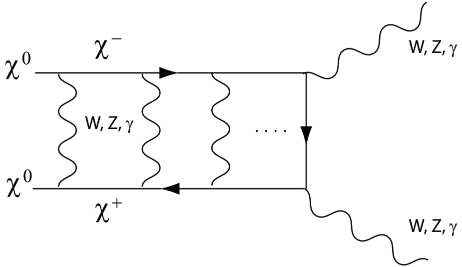
$$\left(\frac{1}{2}\sum |M_1|^2 [1 + f_B(\varepsilon_{p_1})][1 + f_B(\varepsilon_{p_2})] + N_f \sum |M_2|^2 [1 - f_F(\varepsilon_{p_1})][1 - f_F(\varepsilon_{p_2})]\right)$$

Thermal Sommerfeld Effect in QGP

- enhancement of co-annihilation for slowly moving particle :

$$\sim \frac{\alpha}{v} \tag{1}$$

per one rung in a ladder diagram, large for small v



Thermal Sommerfeld Effect in QGP

- Sommerfeld effect enhances the Born matrix elements

$$|\mathcal{M}_{\text{resummed}}|^2 = \mathcal{S}|\mathcal{M}_{\text{tree}}|^2 \quad (1)$$

for the color singlet

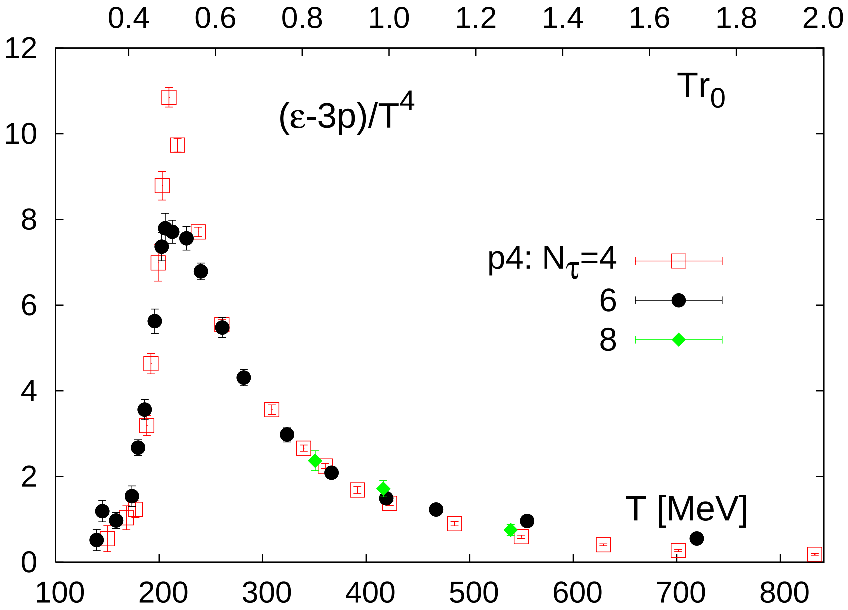
$$\mathcal{S}_1 = \frac{X_1}{1-e^{-X_1}}, X_1 = C_F\frac{g^2}{4v} \quad (2)$$

and for the color octet

$$\mathcal{S}_8 = \frac{X_8}{e^{X_8}-1}, X_8 = (\frac{N_c}{2} - C_F)\frac{g^2}{4v} \quad (3)$$

- the velocity of the initial state has Maxwell distribution
- $\rightarrow$ thermal Sommerfeld effect

$\Gamma_{\rm chem}$ as a transport coefficient



$\Gamma_{\rm chem}$ as a transport coefficient

• we need non-perturbative formulation for the chemical equilibriation rate)

• Langevin equation for the linearized equation (deviation from thermal equilibrium)

$$\partial_t n = -\Gamma_{\rm chem}(n - n_{eq}) + \mathcal{O}(n - n_{eq})^2 \tag{4}$$

where $\Gamma_{\rm chem} = 2cn_{eq}$, chemical equilibriation rate and n is the number density of heavy quark

• linearized form is a Langevin equation and is valid beyond perturbation (D. Bödeker, M. Laine, JHEP07 (2012) 130, JHEP01 (2013) 037)

$\Gamma_{\rm chem}$ as a transport coefficient

- equilibriation rate is a real-time quantity

- lattice gauge theory is a method which can calculate non-perturbative quantities using first principles of quantum field theory

- lattice gauge theory is defined on a Euclidean space and has difficulty in calculating real-time quantity

$\Gamma_{\rm chem}$ as a transport coefficient

- chemical equilibriation as a transport coefficient (D. Bödeker, M. Laine, JHEP07 (2012) 130, 01 (2013) 037)

- treat the approach to the equilibrium as a Langevin process

$$\delta\dot{n}(t) = -\Gamma_{\rm chem}\delta n(t) + \xi(t) \tag{4}$$

$$\langle\langle\xi(t)\xi(t')\rangle\rangle = \Omega_{\rm chem}\delta(t-t'), \qquad \langle\langle\xi(t)\rangle\rangle = 0 \tag{5}$$

where $\delta n(t)$ is the deviation from the equilibrium and $\xi(t)$ is a stochastic noise

$$\delta n(t) = \delta n(t_0)e^{-\Gamma_{\rm chem}(t-t_0)} + \int_{t_0}^{t} dt' e^{\Gamma_{\rm chem}(t'-t)}\xi(t') \tag{6}$$

$\Gamma_{\rm chem}$ as a transport coefficient

- consider a Kubo formula

$$\Delta_{cl}(t,t') = \lim_{t_0\to\infty}\langle\langle\delta n(t)\delta n(t')\rangle\rangle = \frac{\Omega_{\rm chem}}{2\Gamma_{\rm chem}} e^{-\Gamma_{\rm chem}|t-t'|} \tag{4}$$

- from the spectral function from the corresponding Kubo formula

$$\Omega_{\rm chem} = \lim_{\omega\ll T} 2T\omega\rho_{\Delta}(\omega) \tag{5}$$

and

$$\Gamma_{\rm chem} = \frac{\Omega_{\rm chem}}{2\chi_f M^2} \tag{6}$$

$\Gamma_{\rm chem}$ as a transport coefficient

- need to consider, a correlator

$$\int_{\mathbf{x},\mathbf{y}} \langle H(\tau,\mathbf{x})H(0,\mathbf{y})(\theta^\dagger\chi)(\tau_1,\mathbf{0})(\chi^\dagger\theta)(\tau_2,\mathbf{0})\rangle|_{\tau_1>\tau_2}\left[\theta(\tau-\tau_1)+\theta(\tau_2-\tau)\right] \tag{4}$$

$\Gamma_{\rm chem}$ as a transport coefficient

$$P_1 \equiv \frac{1}{2N_c}\mathrm{re},\left\langle G^{\theta}_{\alpha\alpha;ii}(\beta,\vec{0};0,\vec{0})\right\rangle , \tag{4}$$

$$P_2 \equiv \frac{1}{2N_c}\left\langle G^{\theta}_{\alpha\gamma;ij}(\beta,\vec{0};0,\vec{0})G^{\theta\dagger}_{\gamma\alpha;ji}(\beta,\vec{0};0,\vec{0})\right\rangle , \tag{5}$$

$$P_3 \equiv \frac{1}{2N_c^2}\left\langle G^{\theta}_{\alpha\alpha;ij}(\beta,\vec{0};0,\vec{0})G^{\theta\dagger}_{\gamma\gamma;ji}(\beta,\vec{0};0,\vec{0})\right\rangle . \tag{6}$$

- singlet Sommerefeld factor

$$\bar{S}_1 = \frac{P_2}{P_1^2} . \tag{7}$$

- octet Sommerefeld factor

$$\bar{S}_8 = \frac{N_c^2 P_3 - P_2}{(N_c^2-1)P_1^2} . \tag{8}$$

Quarkonium

- bound state of heavy quarks (c,b) (not top quark): $b\bar{b}, c\bar{c}$

- binding through QCD

- analogous to “positronium”

- separation of the “short distance” scale (analytically computable) physics from the “long distance” scale (lattice or phenomenological) physics : M, Mv, Mv^2

- asymptotic freedom and confining force

Hydrogen atom

- quantum mechanics of hydrogen atom

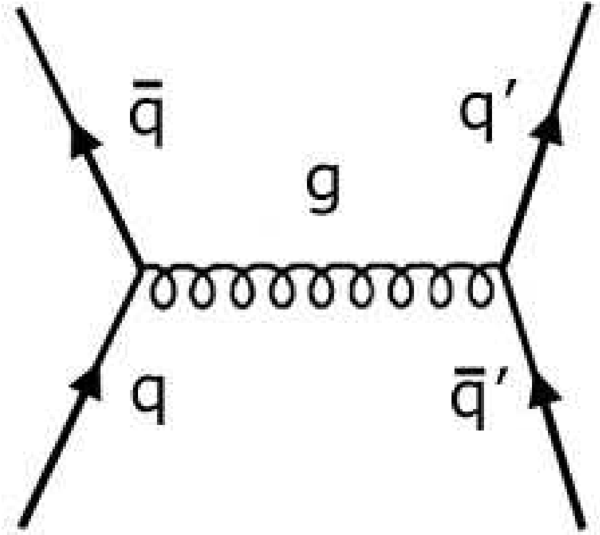
$$i\hbar\frac{\partial\phi}{\partial t} = \mathcal{H}\phi \tag{9}$$

with

$$H = \frac{\mathbf{p}^2}{2m_{\mathrm{red}}} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \cdots \tag{10}$$

- one photon exchange gives a good description for the Coulombic potential

Hydrogen atom



NRQCD for Quarkonium

• relativity + quantum mechanics = quantum field theory

$$\mathcal{L} = \overline{\psi}(\not{D} + m)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{9}$$

• how do we go from a relativistic quantum field theory to quantum mechanics ?

NRQCD for Quarkonium

• do you remember relativistic quantum mechanics? and how to derive hyperfine interaction and etc?

• in Coulomb gauge, Foldy-Wouthuysen-Tani transform

• in field theory language, NRQCD is an effective theory in which the momentum mode higher than the heavy quark mass, M, is integrated away

• need to show power-counting and factorization etc (cf. Braaten, Bodwin, Lepage, Phys. Rev. D51 (1995) 1125)

• inclusive decay rate = partonic decay rate $\times$ the probability for heavy quark to meet anti-heavy quark

NRQCD for Quarkonium

$$\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}, \tag{9}$$

with

$$\mathcal{L}_0 = \psi^\dagger \left(D_\tau - \frac{\mathbf{D}^2}{2M} \right) \psi + \chi^\dagger \left(D_\tau + \frac{\mathbf{D}^2}{2M} \right) \chi, \tag{10}$$

and

$$\begin{aligned} \delta\mathcal{L} = \quad & -\frac{c_1}{8M^3} \left[\psi^\dagger (\mathbf{D}^2)^2 \psi - \chi^\dagger (\mathbf{D}^2)^2 \chi \right] \\ & + c_2 \frac{ig}{8M^2} \left[\psi^\dagger (\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D}) \psi + \chi^\dagger (\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D}) \chi \right] \\ & - c_3 \frac{g}{8M^2} \left[\psi^\dagger \sigma \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}) \psi + \chi^\dagger \sigma \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}) \chi \right] \\ & c_4 \frac{g}{2M} \left[\psi^\dagger \sigma \cdot \mathbf{B} \psi - \chi^\dagger \sigma \cdot \mathbf{B} \chi \right]. \end{aligned} \tag{11}$$

NRQCD for Quarkonium

• NRQCD is an effective field theory:

expansion in v, the heavy quark velocity in heavy quarkonium; separation between fast modes (gluon and light quark) and slow modes (heavy quark)

• includes all the gluons $p < M$

• $\frac{\mu}{M} << 1$, $\frac{T}{M} << 1$

Lattice setup

• anisotropic Euclidean lattices (i.e., the time direction lattice spacing is different from the space direction lattice spacing, $a_s/a_t = 3.5$), $a_s = 0.1227(8)$ fm

• $N_f = 2+1$ light quark flavors ($M_\pi \simeq 400$ MeV, $M_K \simeq 500$ MeV)

• $24^3 \times N_t$ lattices

• T_c = 185 MeV, $a_s M = 2.92$

• lattices used for bottomonium at $T \neq 0$ study (G. Aarts et al, JHEP07 (2014) 097) and electric conductivity of QGP (G. Aarts et al, JHEP-2 (2015) 186)

Lattice setup

$$
\begin{aligned}
G^{\theta}(0,\mathbf{x};\cdot) &= \frac{\delta_{\mathbf{x},\vec{0}}}{a_s^3}\,, \\
G^{\theta}(a_t,\mathbf{x};\cdot) &= \left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n U_0^{\dagger}(0,\mathbf{x})\left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n G^{\theta}(0,\mathbf{x};\cdot)\,, \\
G^{\theta}(\tau+a_t,\mathbf{x};\cdot) &= \left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n U_0^{\dagger}(\tau,\mathbf{x})\left(1-\frac{a_t\mathcal{H}_0}{2n}\right)^n \left(1-a_t\,\delta\mathcal{H}\right) G^{\theta}(\tau,\mathbf{x}
\end{aligned}
$$

where U_0 is a time-direction gauge link. The lowest-order Hamiltonian reads

$$\mathcal{H}_0 = -\frac{\Delta^{(2)}}{2M}\,, \tag{12}$$

where $\Delta^{(2)}$ is a discretized gauge Laplacian.

Lattice setup

The higher order correction is

$$\begin{aligned}\delta\mathcal{H} &= -\frac{(\Delta^{(2)})^2}{8M^3} + \frac{ig_0\,(\nabla\cdot\mathbf{E}-\mathbf{E}\cdot\nabla)}{8M^2} - \frac{g_0\,\sigma\cdot(\nabla\times\mathbf{E}-\mathbf{E}\times\nabla)}{8M^2} \\ &\quad - \frac{g_0\,\sigma\cdot\mathbf{B}}{2M} + \frac{a_s^2\Delta^{(4)}}{24M} - \frac{a_t(\Delta^{(2)})^2}{16nM^2}\,, \end{aligned} \tag{9}$$

Lattice setup

$$P_1 \equiv \frac{1}{2N_c}\mathrm{re}\left\langle G^{\theta}_{\alpha\alpha;ii}(\beta,\vec{0};0,\vec{0})\right\rangle , \quad (9)$$

$$P_2 \equiv \frac{1}{2N_c}\left\langle G^{\theta}_{\alpha\gamma;ij}(\beta,\vec{0};0,\vec{0})G^{\theta\dagger}_{\gamma\alpha;ji}(\beta,\vec{0};0,\vec{0})\right\rangle , \quad (10)$$

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- singlet Sommerefeld factor

$$\bar{S}_1 = \frac{P_2}{P_1^2} . \quad (12)$$

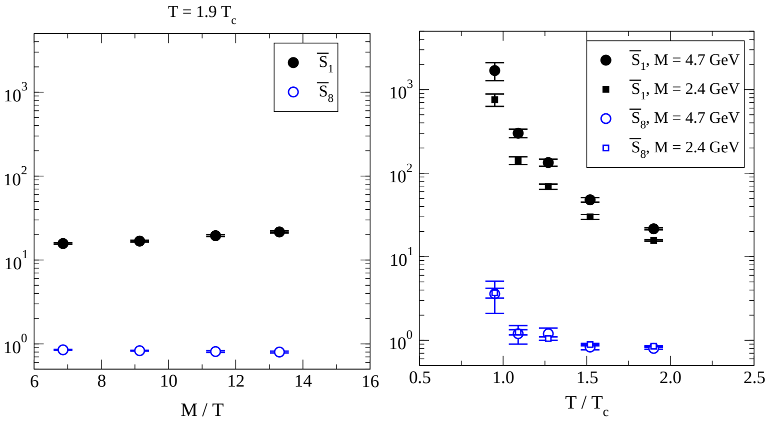
- octet Sommerefeld factor

$$\bar{S}_8 = \frac{N_c^2 P_3 - P_2}{(N_c^2-1)P_1^2} . \quad (13)$$

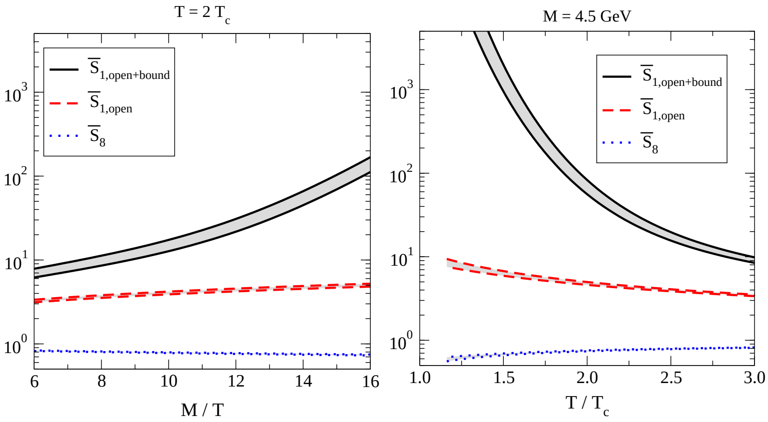
Lattice setup

N_t	T/T_c	$a_s M$	M/T	$10^4 P_1$	$10^5 P_2$	$10^6 P_3$	$\bar{S}_1$	$\bar{S}_8$
32	0.95	2.92	26.7	0.75(7)	0.938(5)	1.06(1)	1690(410)	3.6(15)
28	1.09	2.92	23.4	3.0(1)	2.78(1)	3.19(3)	301(35)	1.2(3)
24	1.27	2.92	20.0	8.0(3)	8.64(3)	10.3(1)	134(13)	1.2(2)
20	1.52	2.92	16.7	24.4(5)	28.5(1)	36.1(4)	48(3)	0.83(6)
16	1.90	2.92	13.3	68.9(9)	102.2(3)	147(1)	21.6(6)	0.80(2)
32	0.95	1.50	13.7	1.31(7)	1.308(8)	1.51(1)	758(128)	3.7(5)
28	1.09	1.50	12.0	4.6(1)	3.04(2)	3.62(3)	142(15)	1.25(9)
24	1.27	1.50	10.3	10.4(2)	7.38(4)	9.21(8)	69(5)	1.05(5)
20	1.52	1.50	8.57	25.7(3)	19.7(1)	27.2(2)	30(2)	0.89(3)
16	1.90	1.50	6.86	61.9(6)	60.1(3)	95.8(8)	15.7(3)	0.85(1)
16	1.90	2.00	9.14	57.6(6)	55.7(2)	86.4(8)	16.8(4)	0.83(1)
16	1.90	2.50	11.4	62.5(7)	76.0(2)	113(1)	19.5(5)	0.81(2)

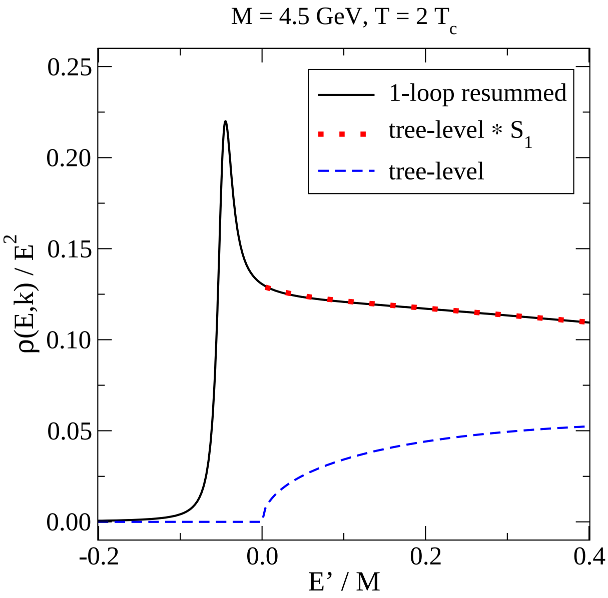
Lattice result of thermal Sommerfeld factor



Analytic estimate of thermal Sommerfeld factor



Analytic estimate of thermal Sommerfeld factor



Conclusion

- a real time quantity, chemical equilibriation rate, is calculated non-perturbatively using Euclidean lattice without analytic continuation

- in the current relativistic heavy ion collision experiments, charm quark and bottom quarks do not chemically equilibriate

- thermal Sommerfeld effect for bottomonium co-annihilation is calculated using lattice NRQCD and is found to be two orders of magnitude larger than perturbative estimate

- bound state effect is the cause of large enhancement

- similarly, in “strongly interacting” dark matter scenario, the bound state effect is expected to enhance co-annihilation of dark matter far beyond a naive perturbative estimate.